

SOME SUPPORTS OF FOURIER TRANSFORMS OF SINGULAR MEASURES ARE NOT RAJCHMAN

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ABSTRACT. The notion of Riesz sets tells us that a support of Fourier transform of a measure with non-trivial singular part has to be large. The notion of Rajchman sets tells us that if the Fourier transform tends to zero at infinity outside a small set, then it tends to zero even on the small set. Here we present a new angle of an old question: Whether every Rajchman set should be Riesz.

1. INTRODUCTION

The consideration of the properties of measures and their Fourier transforms is a classical area of Harmonic Analysis. In particular the following is well known.

Theorem 1.1 (Rajchman, 1929 [4]). *If for a finite measure μ on the unit circle \mathbb{T} holds $\widehat{\mu}(n) \rightarrow 0$ when $n \rightarrow -\infty$, then it holds also that $\widehat{\mu}(n) \rightarrow 0$ when $n \rightarrow +\infty$.*

This motivates the following.

Definition 1.2. We say that $\Lambda \subset \mathbb{Z}$ is a *Rajchman set* if as soon as $\widehat{\mu}(n) \rightarrow 0$ when $|n| \rightarrow +\infty, n \in \mathbb{Z} \setminus \Lambda$, then $\widehat{\mu}(n) \rightarrow 0$ when $|n| \rightarrow +\infty, n \in \Lambda$.

With this definition the Rajchman theorem says that the non-negative integers is a Rajchman set.

Now, given a (signed) Radon measure μ on the unit circle \mathbb{T} , we can present it as $\mu = f \cdot m + \mu_s$, where m is the Lebesgue measure and μ_s is the singular with respect to Lebesgue measure part of the measure μ . We known the following.

Theorem 1.3 (F. and M. Riesz's, 1916, [5]). *If a finite measure μ has the property $\widehat{\mu}(-n) = 0$ for $n = 1, \dots$, then the measure is absolutely continuous with respect to Lebesgue measure, i.e. $\mu = f \cdot m$, where $f \in L^1(\mathbb{T})$.*

This result motivates the following definition.

Definition 1.4. We say that a subset $\Lambda \subset \mathbb{Z}$ is a Riesz set if it has the property, that if $\text{supp}(\widehat{\mu}) \subset \Lambda$ then μ has no singular part.

With this definition the F. and M. Riesz theorem says that the non-negative integers is a Riesz set.

Theorem 1.5 (Host, Parreau, 1978 [1]¹). *A set $\Lambda \subset \mathbb{Z}$ is a Rajchman set iff it doesn't contain any shift of the Fourier support of a Riesz product, i.e. any set $\Omega((n_j)) = \{\sum \epsilon_j n_j : \epsilon_j = -1, 0, 1; \sum |\epsilon_j| < \infty\}$, where (n_j) is an infinite sequence.*

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¹It is actually proven in [1] not only for \mathbb{T} but for any compact group

Thus, any set which is not Rajchman, contains the support of the Fourier transform of a singular measure, and thus is not Riesz (or, without negations, that every Riesz set is a Rajchman set).

A natural question is following: Is every Rajchman set a Riesz set? (I.e. Do the classes of Riesz and Rajchman sets coincide?) As far to the author's knowledge, this question was first raised by Pigno, 1978 [3].

As we are unable to answer the question, we want to diversify it:

Definition 1.6. We say that a closed set $E \subset \mathbb{T}$ is a *parisian set* if for every non absolutely continuous measure $\mu \in M(E)$, the support of it's Fourier transform is not a Rajchman set.

The original question thus becomes: Is \mathbb{T} a parisian set?

While we are not able to answer the question above, we can show that some parisian sets do exist. As any subset of a parisian set is parisian, it is clear that a positive answer on the original question would imply all the results we prove here. Yet, there are good chances that the answer is negative and a negative answer would give the study of the parisian sets some interest.

It is natural to expect that the parisian sets should be "small". Thus we try to construct a "big" parisian set.

Main Theorem A. *For any $\alpha < 1$ there exists a closed parisian set E , such that $\dim_H(E) \geq \alpha$, where $\dim_H(E)$ means the Hausdorff dimension of E .*

Main Theorem B. *For any $\alpha < 1$ there exists a Borel parisian set E such that it is an additive subgroup of \mathbb{T} and $\dim_H(E) \geq \alpha$.*

Notations. In what follows we identify \mathbb{T} with $(-1, 1]$, so that the Fourier coefficients are $\hat{\mu}(n) = \frac{1}{2} \int e^{i\pi n x} d\mu(x)$.

2. CONSTRUCTION OF A BIG PARISIAN SET

Let us first introduce a test to establish that a set is parisian.

Lemma 2.1. *If there exist $\delta > 0$ and a sequence $(N_j)_{j=1}^\infty$ such that for every j the set E is a subset of $\frac{2}{N_j}\mathbb{Z} + [-1/2N_j^{1+\delta}, 1/2N_j^{1+\delta}]$, then the set E is parisian.*

Proof. Let us fix $\mu \in M_s(E)$. We want to show that $\text{supp}(\hat{\mu})$ contains a shift of a set $\Omega((n_j))$. Up to a shift of the Fourier transform we may assume without loss of generality that $\hat{\mu}(0) \neq 0$.

Here we construct the sequence (n_j) as a subsequence of (N_j) inductively. Assume that $(k-1)$ first terms of the sequence (n_j) are chosen. This means that for all combinations of ϵ_j the sum $\sum_{j=1}^{k-1} \epsilon_j n_j \in \text{supp}(\hat{\mu})$. Thus, we know that

$\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} d\mu(x) \neq 0$, for all combinations $(\epsilon_j = -1, 0, 1)_{j=1}^{k-1}$. We can take γ_{k-1} to be the minimum of the absolute value of the 3^{k-1} non-zero numbers, so that

$|\int e^{i\pi \sum_{j=1}^{k-1} \epsilon_j n_j x} d\mu(x)| \geq \gamma_{k-1}$. We want to show that for some sufficiently large

$n_k = N_{j_k}$ for all combinations of ϵ_j holds $\int e^{i\pi \sum_{j=1}^k \epsilon_j n_j x} d\mu(x) \neq 0$.

Indeed, as $E \subset 2\mathbb{Z}/N_m + [-1/N_m^{1+\delta}, 1/N_m^{1+\delta}]$, we know that $|e^{i\pi(\pm N_m x)} - 1| \leq \frac{\pi}{N_m^\delta}$, when $x \in E$. Now we see that

$$\left| \int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} d\mu(x) - \int_E e^{i\pi \sum_{j=1}^{k-1} \varepsilon_j n_j x} d\mu(x) \right| \leq \int_E |d\mu| |e^{i\pi \pm N_m x} - 1| \leq \|\mu\| \frac{1}{N_m^\delta}.$$

Thus, for sufficiently large m we can be sure that the later is less than $\frac{1}{2}\gamma_{k-1}$. Now,

we see that by the triangle inequality $|\int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} d\mu(x)| \geq \frac{1}{2}\gamma_{k-1} > 0$ for all the combinations of $\varepsilon_j = -1, 0, 1$, with $j = 1, \dots, k$, and $n_k = N_m$. \square

A slight modification of the proof gives us the following.

Lemma 2.2. *For an increasing sequence $(N_j) \subset \mathbb{N}$ and $\delta > 0$ the set $\tilde{E} = \{x \in \mathbb{T} : \sup_j \text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta} < \infty\}$ is a parisian set.*

Proof. We start from observing that $\tilde{E} = \bigcup_{t \in \mathbb{N}} E_t$, where

$$E_t = \{x \in \mathbb{T} : \sup_j \text{dist}(x, 2\mathbb{Z}/N_j)/N_j^{1+\delta} \leq t\}$$

is an increasing sequence of closed sets.

Now, we start the proof exactly as the previous one, but after the choice of γ_{k-1} and before the choice of n_k we do one more step: We pick t_k large enough that $\mu_k =$

$\mu|_{E_k}$ satisfies $\|\mu - \mu_k\| < \frac{1}{3}\gamma_{k-1}$. Then we see that $|\int e^{i\pi \sum_{j=1}^{k-1} \varepsilon_j n_j x} d\mu_k(x)| \geq \frac{2}{3}\gamma_{k-1}$. We proceed in the same way as before with μ_k in place of μ , and find $n_k = N_{m_k}$ such that $|\int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} d\mu_k(x)| \geq \frac{1}{2}\gamma_{k-1}$. Then, $|\int_E e^{i\pi \sum_{j=1}^k \varepsilon_j n_j x} d\mu(x)| \geq \frac{1}{6}\gamma_{k-1} > 0$. \square

Remark 2.3. The set \tilde{E} is obviously an additive subgroup of \mathbb{T} and thus either finite or dense in \mathbb{T} .

Let us now construct a set E of large Hausdorff dimension which satisfies the hypothesis of the Lemma 2.1, and is thus parisian. As the constructed set is a subset of \tilde{E} it will also give us the estimate² on the Hausdorff dimension of \tilde{E} . Fix $\alpha \in (0, 1)$, and choose $\delta > 0$ so that $\delta = 1 - \alpha$. We will construct a rapidly increasing sequence $\{N_j\}$, and related sequence of closed sets $C_j \subset (-1, 1)$, such that the sets C_j is the union of the closed intervals with centurms in $2\mathbb{Z}/N_j$, of length $1/N_j^{1+\delta}$ which are entirely contained in $\bigcap_{k=1}^{j-1} C_k$. We will let then the set $E = \bigcap_j C_j$, which is obviously closed. The set constructed in such a way is a Cantor-type set, and we show that provided the sequence N_j grows quickly enough the dimension of such a set is at least α .

Lemma 2.4. $\dim_H(E) \geq \alpha$.

Proof. In order to prove that the Hausdorff dimension of E is at least α we will show that it is at least s for any $0 < s < \alpha$, and to do so we construct a finite measure μ supported on E such that $\mu(I) \leq c_s |I|^s$ for any interval I (it is a standard fact of

²This estimate is well known, but we give the proof for the sake of completeness.

Geometric Measure Theory that a measure satisfying such an estimate should have support of Hausdorff dimension at least s , see for example [2]).

Let us take a subset D_k of $\bigcap_{j=1}^k C_j$, which is a collection of intervals of length $1/N_k^{1+\delta}$. This collection is defined inductively: we know that every interval of length $1/N_{k-1}^{1+\delta}$ contains at least $N_k/2N_{k-1}^{1+\delta} - 1$ points of $2\mathbb{Z}/N_k$. Thus, every interval of D_{k-1} contains (entirely) at least $M_k = N_k/2N_{k-1}^{1+\delta} - 3$ intervals with centrum in $2\mathbb{Z}/N_k$ and length $1/N_k^{1+\delta}$. (To make the estimates more simple we assume (N_k) to grow so rapidly that $M_k \geq N_k/4N_{k-1}^{1+\delta}$.)

We pick from each interval of D_{k-1} exactly M_k such intervals. All together we will have picked $M_k \prod_{j=1}^{k-1} M_j$ intervals of length $\frac{1}{N_k^{1+\delta}}$. Then we take the probability

measure μ_k equally distributed on the $\prod_{j=1}^k M_j$ intervals of D_k . We introduce μ as a weak limit point of μ_k (which has to be a probability measure supported by $E = \bigcap C_j$).

Let us estimate $\mu(I)$ where $1/N_{k-1} > |I| \geq 1/N_k$. The interval can intersect at most $N_k|I|/2 + 3$ intervals of D_k (as $N_k|I| \geq 1$, we may use that it is at most $4N_k|I|$ intervals). As the measure of each interval of D_k is $1/\prod_{j=1}^k M_j$ we see that $\mu(I) \leq 4|I|N_k/\prod_{j=1}^k M_k$ where

$$\prod M_k \geq (N_k/N_1)/(4^{k-1}(\prod_{j=1}^{k-1} N_j)^\delta).$$

Thus, $\mu(I) \leq N_1 4^k (\prod_{j=1}^{k-1} N_j)^\delta |I| = N_1 4^k (\prod_{j=1}^{k-1} N_k)^\delta |I|^{1-s} |I|^s$.

Our task is fulfilled if we show that $c_{k,s} = N_1 4^k (\prod_{j=1}^{k-1} N_j)^\delta |I|^{1-s}$ is bounded above independently from k . We know that $|I| < 1/N_{k-1}$, and, as $\delta = 1 - \alpha$, we see that $c_{k,s} \leq N_1 4^k (\prod_{j=1}^{k-2} N_j)^\delta / N_{k-1}^{\alpha-s}$. It remains to take the sequence (N_k) such that $(N_1 4^{k+2} (\prod_{j=1}^k N_j)^\delta)^k < N_{k+1}$. For any fixed s the sequence $c_{k,s}$ tends to zero, and so is bounded. (Notice that the bound $c_s = \sup_k \{c_{k,s}\}$ grows as $s \rightarrow \alpha$, but we only need it to be finite.) \square

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